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Characteristic Exponents of Impulsive Differential Equations in a Banach Space

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The notion of general exponent of impulsive homogeneous differential equations is defined. A formula for the solution of impulsive nonhomogeneous differential equations is obtained and is used to establish a dependence between the existence of bounded solutions of such equations and the general exponent of the respective homogeneous equation.

1. INTRODUCTION

Impulsive differential equations are used to study dynamical processes that are subject to short-time perturbations during their evolution. The duration of these perturbations is negligibly small; that is why they are considered momentary, i.e., the perturbations are in the form of impulses. Dynamical processes with such perturbations are studied, e.g., in physics, chemistry, and control theory.

The qualitative theory of impulsive equations was originated by Millman and Myshkis (1960) and was further developed by Myshkis and Samoilenko (1967), Samoilenko and Perestiuk (1977), and Simeonov and Bainov (1985a,b). Zabreiko *et al.* (to appear) marks the beginning of the investigation of impulsive equations in a Banach space. In the present paper, following the ideas of Daleckii and Krein (1974), a formula for the solution of the nonhomogeneous impulsive linear equation is derived and is used to establish a dependence between the existence of bounded solutions of this equation and the general exponent of the respective homogeneous equation.

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2. STATEMENT OF THE PROBLEM

Consider the following impulsive differential equation:

$$\frac{dx}{dt} = A(t)x\big|_{t \neq t_n} \tag{1}$$

$$x(t_n + 0) = Q_n x(t_n - 0) \tag{2}$$

where $A(t)$ $(t \geq t_0)$ and Q_n $(n = 1, 2, \ldots)$ are linear, bounded operators mapping the complex Banach space X into itself, and t_n $(n = 1, 2, \ldots)$ are fixed impulsive moments satisfying the condition

$$0 = t_0 < t_1 < t_2 < \cdots, \qquad \lim_{n \to \infty} t_n = \infty$$

Definition 1. We shall call a solution of the impulsive equation (1), (2) a piecewise continuous function $x(t)$ with points of discontinuity of first type t_1, t_2, ... such that

$$\frac{dx}{dt} = A(t)x(t) \tag{3}$$

for $t \neq t_n$ and

$$x(t_n + 0) = Q_n x(t_n - 0) \qquad (n = 1, 2, \cdots)$$

Assume that at the points of discontinuity the function is left continuous. Denote by $U(t, \tau)$ the evolutionary operator of equation (1). For any $x_0 \in X$ the impulsive equation (1), (2) has a unique solution $x(t)$ satisfying the condition

$$x(t_0) = x_0 \tag{4}$$

Consider the operator-valued function $V(t)$ defined by the formula

$$V(t)x_0 = x(t) \qquad (t_0 \leq t < \infty)$$

where $x(t)$ is the solution of (1), (2) with initial condition (4).

Lemma 1. Let the operators Q_n $(n = 1, 2, \ldots)$ be invertible. Then, for $t_n \leq t < t_{n+1}$ $(n = 0, 1, \ldots)$ the following equality holds:

$$V(t) = U(t, t_n) Q_n U(t_n, t_{n-1}) Q_{n-1} \cdots Q_1 U(t_1, t_0) \tag{5}$$

and the evolutionary operator of the impulsive equation (1), (2) has the form

$$W(t,\tau)=\begin{cases} U(t,t_n)\left[\prod_{j=n}^{k+1} Q_jU(t_j,t_{j-1})\right]Q_kU(t_k,\tau) \\ \quad (t_{k-1}<\tau\leq t_k<t_n<t\leq t_{n+1}) \\ U(t,t_n)\left[\prod_{j=n}^{k-1} Q_jU(t_j,t_{j+1})\right]Q_k^{-1}U(\tau_k,s) \\ \quad (t_{n-1}<t\leqslant t_n<t_k<s\leq t_{k+1}) \end{cases} \tag{6}$$

Lemma 1 is proved by straightforward verification.

Definition 2. A general exponent κ_g of the impulsive equation (1), (2) we shall call the greatest lower bound of all numbers ρ such that for each solution $x(t)=V(t)x_0$ $(x_0\in X)$ of the impulsive equation (1), (2) the inequality

$$\|x(t)\|\leq N_\rho\, e^{\rho(t-\tau)}\|x(\tau)\|$$

holds, where the number N_ρ does not depend on the choice of x_0.

3. MAIN RESULTS

We shall give conditions under which the general exponent of the impulsive equation is finite. We shall obtain a fundamental formula for the solution of a nonhomogeneous impulsive differential equation and show the relation between the existence of bounded solutions of such equations and the general exponent of the impulsive equation (1), (2).

Theorem 1. Let the operators $Q_n (n=1,2,\ldots)$ be invertible. Then a necessary and sufficient condition for the general exponent κ_g of the impulsive equation (1), (2) to be a finite number is the existence of $T>0$ satisfying the inequality

$$K_T=\sup_{0\leq t-\tau\leq T}\|W(t,\tau)\|<\infty \tag{7}$$

Proof. The proof of Theorem 1 is a modification of the proof of Theorem 4.2 of Daleckii and Krein (1974). ■

Corollary 1. Condition (7) implies the estimate

$$\kappa_g\leq T^{-1}\ln\sup_{0\leq t-\tau\leq T}\|W(t,\tau)\| \tag{8}$$

Remark 1. $\kappa_g<\infty$ implies inequality (7) for any $T>0$.

Definition 3. The operator-valued function $A(t)$ $(t\geq t_0)$ is called integral-bounded with a constant M if for $t\geq t_0$ the following inequality holds:

$$\int_t^{t+1}\|A(\tau)\|\,d\tau\leq M \tag{9}$$

Theorem 2. Let the following conditions hold:

1. The operators $Q_n (n=1, 2, \ldots)$ are invertible.
2. The operator-valued function $A(t)$ is integral-bounded with a constant M.
3. $\sup_{0 \le t-\tau \le T} \prod_{\tau \le t_n < t} \|Q_n\| \le \Delta$ for $\tau \ge t_0$, where $T > 0$, $\Delta \ge 0$ are constants.

Then $\kappa_g < \infty$.

Proof. In view of (8) and Lemma 1, we obtain the inequalities

$$\begin{aligned}
\kappa_g &\le T^{-1} \ln \sup_{0 \le t-\tau \le T} \|U(t, t_n)Q_n \cdots Q_{k+1} U(t_{k+1}, t_k) Q_k U(t_k, \tau)\| \\
&\le T^{-1} \ln \sup_{0 \le t-\tau \le T} \left\{ \exp\left[\int_{t_n}^{t} \|A(s)\| \, ds \right] \exp\left[\int_{t_{n-1}}^{t_n} A(s)\| \, ds \right] \cdots \right. \\
&\quad \times \exp\left[\int_{t_k}^{t_{k+1}} \|A(s)\| \, ds \right] \exp\left[\int_{\tau}^{t_k} \|A(s)\| \, ds \right] \\
&\quad \left. \times \|Q_n\| \cdots \|Q_k\| \right\} \\
&= T^{-1} \ln \sup_{0 \le t-\tau \le 1} \left\{ \exp\left[\int_{\tau}^{t} \|A(s)\| \, ds \right] \|Q_n\| \cdots \|Q_k\| \right\} \\
&\le T^{-1} \ln \sup_{0 \le t-\tau \le T} \{\exp[M(2+T)] \, \|Q_n\| \cdots \|Q_k\|\} \\
&\le T^{-1} \ln\{\exp[M(2+T)]\Delta\} < \infty
\end{aligned}$$

Theorem 2 is proved. ■

Consider the impulsive equation

$$\frac{dx}{dt} = A(t)x + f(t) \,\Big|_{t \ne t_n} \tag{10}$$

$$x(t_n + 0) = Q_n x(t_n) \qquad (n = 1, 2, \ldots) \tag{11}$$

where the function $f(t)$ $(t \ge t_0)$ assumes values in X.

Lemma 2. Let the operators Q_n $(n=1, 2, \ldots)$ be invertible. Then the solution of the Cauchy problem for the impulsive equation (10), (11) with initial condition (4) is given by the fomula

$$x(t) = W(t, t_0)x_0 + \int_{t_0}^{t} W(t, \tau) f(\tau) \, d\tau \tag{12}$$

Lemma 2 is proved using standard methods.

Theorem 3. Let the operators Q_n $(n=1, 2, \ldots)$ be invertible. Let, moreover,

$$x(0) = 0 \tag{13}$$

and to any function $f(t)$ continuous and bounded on $[0, \infty)$ there corresponds a bounded solution $x(t)$ of the Cauchy problem for the impulsive equation (10), (11) and initial condition (13).

Then there exist constants N, $\nu > 0$ such that for $t \geq 0$ the following inequality holds:

$$\| V(t) \| \leq N e^{-\nu t} \tag{14}$$

Proof. By Lemma 2 the solution of the Cauchy problem for the impulsive equation (1), (2) and initial condition (13) has the form

$$x(t) = \int_0^t W(t, \tau) f(\tau) \, d\tau$$

Consider the space $C(X)$ of all continuous and bounded functions $g(t)$ $(0 \leq t < \infty)$ with values in X and norm

$$\|g\|_C = \sup_{0 \leq t < \infty} \|g(t)\| \tag{15}$$

For any t fixed, consider the linear operator V_t: $C(X) \to X$ defined by the formula

$$V_t(f) = \int_0^t W(t, \tau) f(\tau) \, d\tau$$

From the estimate

$$\| V_t(f) \| \leq \int_0^t \| W(t, \tau) \| \cdot d\tau \cdot \|f\|_C$$

it follows that the operator V_t $(t \geq 0)$ is bounded. In virtue of the conditions of Theorem 3 the family of operators V_t $(t \geq 0)$ is uniformly bounded for each $f \in C(X)$. Then by the theorem of Banach-Steinhaus there exists a constant $k > 0$ such that for $t \geq 0$ the following inequality holds:

$$\|x(t)\| = \| V_t(f) \| \leq k \cdot \|f\|_C \tag{16}$$

Set $\chi(t) = \| V(t) \|$ and consider the function $f(t) = [V(t)/\| V(t) \|]y$, where y is an arbitrary element of X. From the inequality $\|f\|_C \leq \|y\|$ it follows that the function f belongs to $C(X)$. The solution $x(t)$ of the impulsive equation (10), (11) corresponding to $f(t)$ is represented by the formula

$$x(t) = \int_0^t W(t, \tau) \frac{V(\tau)}{\chi(\tau)} y \, d\tau = \int_0^t \frac{V(t)}{\chi(\tau)} y \, d\tau = V(t) y \varphi(t)$$

where

$$\varphi(t) = \int_0^t \frac{d\tau}{\chi(\tau)}$$

From (16) we deduce the estimate $[\|V(t)y\|/\|y\|]\varphi(t)\le k$, which implies the inequality $\varphi'(t)/\varphi(t)\ge 1/k(t\ne t_n)$. Since the function $\varphi(t)$ is absolutely continuous, integrating the last inequality from 1 to t gives

$$\varphi(t)\ge\varphi(1)\,e^{(t-1)/k}$$

i.e.

$$\frac{1}{\chi(t)}=\varphi'(t)\ge\frac{\varphi(t)}{k}\ge\frac{\varphi(1)}{k}\,e^{(t-1)/k}$$

We set $\nu=1/k$, $N_1=k\,e^{1/k}\varphi(1)$ and obtain

$$\|V(t)\|=\chi(t)\le N_1\,e^{-\nu t}\qquad(t\ge 1)$$

whence it follows that

$$\|V(t)\|\le N e^{-\nu t}\qquad(t\ge 0),$$

where

$$N=\max(N_1,\max_{0\le t\le 1}e^{\nu t}\|V(t)\|)$$

This completes the proof of Theorem 3. ■

Theorem 4. Let the operators Q_n $(n=1,2,\ldots)$ be the invertible and let the general exponent κ_g of the impulsive equation (1), (2) be negative.

Then for each bounded and continuous function $f(t)$ the Cauchy problem for the impulsive equation (10), (11) with initial condition (13) has a solution which is bounded on the semiaxis $t\ge 0$.

Proof. The solution of the Cauchy problem for the impulsive equation (10), (11) and initial condition (13) has the form

$$x(t)=\int_{t_0}^{t}W(t,\tau)f(\tau)\,d\tau$$

In view of the condition of Theorem 4, there exist constants N, $\nu>0$ such that for $0\le\tau\le t<\infty$ the following inequality $\|W(t,\tau)\|\le N e^{-\nu(t-\tau)}$ holds. For $\|x(t)\|$ we obtain the estimate

$$\begin{aligned}\|x(t)\|&\le\int_{t_0}^{t}\|W(t,\tau)\|\cdot\|f(\tau)\|\,d\tau\\&\le N\int_{t_0}^{t}e^{-\nu(t-\tau)}\,d\tau\cdot\|f\|_C\\&\le\frac{N}{\nu}\|f\|_C(1-e^{-\nu t})\\&\le\frac{N}{\nu}\|f\|_C<\infty\end{aligned}$$

Theorem 4 is proved. ■

We shall find conditions under which the general exponent of the impulsive equation (1), (2) is negative.

Theorem 5. Let the following conditions be fulfilled:

1. The operators $Q_n (n=1,2,\ldots)$ are invertible.
2. The conditions of Theorem 2 hold.
3. For each function $f \in C(X)$ the Cauchy problem for the impulsive equation (10), (11) and initial condition (13) has a solution which is bounded for $t \geq 0$.

Then the general exponent of the impulsive equation (1), (2) is negative.

Proof. The boundedness of the solution of the Cauchy problem for the impulsive equation (10), (11) with initial condition (13) implies the boundedness of the solution of the problem

$$\frac{dx}{dt}=A(t)x+f(t), \qquad t \geq t^* > 0, \quad t \neq t_n \tag{17}$$

$$x(t_n+0)=Q_n x(t_n), \qquad t \neq t_n \tag{18}$$

for $n-1,2,\ldots,$ with initial condition

$$x(t^*)=0 \tag{19}$$

In fact, let $t_{m-1} \leq t^* < t_m$. Then the solution of problem (17)-(19) is given by the formula

$$x(t)=\int_{t^*}^{t} W(t,\tau)f(\tau)\,d\tau \tag{20}$$

where the function $W(t,\tau)$ is defined by equality (6). Consider the Cauchy problem

$$\frac{dx_\varepsilon(t)}{dt}=A(t)x_\varepsilon(t)+f(t) \qquad \text{for } t \geq 0, \quad t \neq t_n \tag{21}$$

$$x_\varepsilon(t_n+0)=Q_n x_\varepsilon(t_n), \qquad n=1,2,\ldots \tag{22}$$

with initial condition

$$x_\varepsilon(0)=0 \tag{23}$$

where

$$f_\varepsilon(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < t^* - \varepsilon \\ \dfrac{1}{\varepsilon} f(t^*)(t - t^* + \varepsilon) & \text{for} \quad t^* - \varepsilon \le t \le t^* \\ f(t) & \text{for} \quad t \ge t^* \end{cases}$$

The solution of problem (21)-(23) is represented in the form

$$x_\varepsilon(t) = \int_{t^*-\varepsilon}^{t^*} W(t, \tau) f_\varepsilon(\tau)\, d\tau + \int_{t^*}^{t} W(t, \tau) f(\tau)\, d\tau$$

As in the proof of Theorem 3, a constant k can be found such that the estimate $\|x_\varepsilon(t)\| \le k \cdot \|f\|_C$ holds. Following the scheme of the proof of Theorem 3 with minor modifications, we obtain the estimates

$$\left\| \int_{t^*}^{t} W(t, \tau) f(\tau)\, d\tau \right\| \le k \|f\|_C$$

$$\| W(t, t^*) \| \le N e^{-\nu(t-t^*)}$$

where $\nu = 1/k$, $N \ge \max\{k\, e^{1/k}/\varphi(T), P\}$, and

$$\varphi(T) = \int_{t^*}^{t^*+T} \frac{d\tau}{\| W(t, t^*)\|}, \qquad P = \max_{0 \le t - t^* \le T} e^{\nu(t-t^*)} \| W(t, t^*)\|$$

We shall show that N can be chosen independent of t^*. In fact, for $t \in (t^*, t^* + T]$ the following estimates hold:

$$\begin{aligned} \varphi(T) &\ge \int_{t^*}^{t^*+T} e^{-M(T+2)} \Delta^{-1}\, d\tau = \frac{e^{-M(T+2)}}{\Delta} T \\ &\le \max_{t \in [t^*, t^*+T]} \{ e^{\nu(t-t^*)} \| W(t, t^*) \| \} \\ &\le \max_{t \in [t^*, t^*+T]} e^{\nu(t-t^*) + M(T+2)} \Delta \\ &\le e^{T\nu + M(T+2)} \Delta \end{aligned}$$

We set

$$N = e^{1/k + M(T+2)} \Delta \max\{k/T, e^{T-1}\}$$

and obtain

$$\| W(t, t^*) \| \le N e^{-\nu(t-t^*)} \qquad (t^* > 0)$$

i.e., for $0 \le \tau \le t < \infty$ the following inequality holds: $\| W(t, \tau) \| \le N e^{-\nu(t-\tau)}$.

Theorem 5 is proved. ■

Theorem 6. Let the operators Q_n $(n=1,2,\ldots)$ be invertible and $\kappa_g<\infty$. Then a necessary and sufficient condition for $\kappa_g<0$ is the existence of a positive constant T and of a constant $q\in(0,1)$ such that for $x\in X$ and $t\geq 0$ there exists a number $\theta_{x,t}\in[0,T]$ satisfying the inequality

$$\|W(t+\theta_{x,t},t)\|\leq q\cdot\|x\| \tag{24}$$

Proof. The necessity is proved in a trivial way. We shall prove the sufficiency. Let $0\leq\bar{t}<t<\infty$. Let k and l be indices such that $\bar{t}\in(t_k,t_{k+1}]$ and $2t\in(t_l,t_{l+1}]$ and denote $R=\max\{\max_{j=k,\ldots,l}\|Q_j\|,1\}$. For any $\tau,\tau'\in(\bar{t},2t]$ there exists a number θ' such that for $|\tau-\tau'|\leq\theta'$ there exists an index i satisfying the double inequality $t_{i-1}\leq\tau,\tau'\leq t_{i+1}$. The continuity of the operator $U(t,s)$ implies the existence of a number θ'' such that for $|\tau-\tau'|\leq\theta''$ and $\tau,\tau'\in[\bar{t},2t)$ the estmate $\|U(\tau,\tau')\|\leq 1/q^{1/2}R^{1/2}$. We set $\theta=\min\{\theta',\theta''\}$ and obtain that for all $\tau,\tau'\in[\bar{t},2t)$ satisfying the condition $|\tau-\tau'|\leq\theta$ and $\tau'<t_i\leq\tau$ for some index $i=k,\ldots,l$ the following estimate holds:

$$\begin{aligned}\|W(\tau,\tau')\|&=\|U(\tau,t_i)Q_iU(t_i,\tau')\|\\&\leq\|U(\tau,t_i)\|\cdot\|Q_i\|\cdot\|U(t_i,\tau')\|\\&\leq\frac{1}{q^{1/2}R^{1/2}}R\frac{1}{q^{1/2}R^{1/2}}=\frac{1}{q}\end{aligned}$$

When the interval $(\tau',\tau]$ contains no member of the sequence $\{t_n\}$ we again obtain the estimate

$$\|W(\tau,\tau')\|=\|U(\tau,\tau')\|\leq\frac{1}{q^{1/2}R^{1/2}}\leq\frac{1}{q}$$

On the other hand, since $\kappa_g<\infty$, then, by Theorem 1, the inequality $\sup_{0\leq t-\tau\leq T}\|W(t,\tau)\|<\infty$ is satisfied.

Further, the proof of Theorem 6 is a modification of the proof of Theorem 6.1 of Daleckii and Krein (1974). ■

Theorem 7. Let the operators Q_n $(n=1,2,\ldots)$ be invertible and let p be a positive number. Then the general exponent κ_g is negative if and only if there exists a positive constant c such that for $t^*\leq\tau<\infty$ the inequality

$$\int_\tau^\infty\|W(t,\tau)x\|^p\,dt\}^{1/p}\leq C\|x\|$$

holds, where $t^*>0$ is some fixed point.

The proof of Theorem 7 is analogous to the proof of Theorem 6.2 of Daleckii and Krein (1974).

Definition 4 (Daleckii and Krein, 1974). $C: X \to X$ is called ω-limiting for the operator-valued function $A(t)$ $(t \geq t_0)$ if there exists a sequence $\xi_n \to_{n\to\infty} \infty$ such that $A(\xi_n) \to_{n\to\infty} C$.

Definition 5 (Daleckii and Krein, 1974). The operator-valued function $A(t)$ $(t \geq t_0)$ satisfies condition $S_{\varepsilon,L}$ if for some $\varepsilon > 0$ and $L > 0$ there exists a number $T > 0$ such that for $s, t \geq T$ and $|s-t| \leq L$ the inequality $\|A(s) - A(t)\| \leq \varepsilon$ holds.

Definition 6 (Daleckii and Krein, 1974). The operator-valued function $A(t)$ $(t \geq t_0)$ is compact if from any sequence $\{A(t_n)\}$ a subsequence convergent to some linear bounded operator mapping X into X can be chosen.

We shall need the following lemma.

Lemma 3. Let $\varphi(t)$ $(t \geq \tau_0)$ be a nonnegative, piecewise left-continuous function with points of discontinuity of first type, let $h(t)$ $(t \geq \tau_0)$ be a continuous, nonnegative function, and let $c \geq 0$ be a constant.

Let the following inequality hold:

$$\varphi(t) \leq c + \int_{\tau_0}^{t} h(\tau)\varphi(\tau)\, d\tau$$

Then

$$\varphi(t) \leq c \exp\left[\int_{\tau_0}^{t} h(\tau)\, d\tau\right]$$

Proof. Consider the operator K acting in the space $D(R^{\tau_0}, X)$ of the piecewise, left-continuous functions with points of discontinuity of first type and with values in X and defined by the equality

$$(K\varphi)(t) = \int_{\tau_0}^{t} h(\tau)\varphi(\tau)\, d\tau$$

The operator K is of Volterra type; hence, its spectral radius is equal to zero. Then the following estimate holds:

$$\varphi(t) \leq \psi(t)$$

where $\psi(t)$ is a solution of the integral equation

$$\psi(t) = c + \int_{\tau_0}^{t} h(\tau)\psi(\tau)\, d\tau$$

A straightforward verification shows that

$$\psi(t) = c \exp\left[\int_{\tau_0}^{t} h(\tau)\, d\tau\right]$$

Lemma 3 is proved. ■

Lemma 4. Let $W_k(t, s)$ $(k=1,2))$ $(a \le s, t \le b)$ be the respective evolutionary operators of impulsive equations

$$\frac{dx}{dt} = A_k(t)x\big|_{t \neq t_n}$$

$$x(t_n+0) = Q_n x(t_n), \qquad n = 1, 2, \ldots; \quad k = 1, 2$$

If there exist numbers $N > 0$ and $\nu_1 \in (-\infty, \infty)$ such that the inequality

$$\| W_1(t, s) \| \le N e^{-\nu_1(t-s)} \quad (a \le s, t \le b)$$

holds, then the following inequality holds as well:

$$\| W_2(t, s) \| \le N \exp[-\nu_1(t-s)] \times \exp\left[N \int_s^t \|A_1(\tau) - A_2(\tau)\| \, d\tau \right] \qquad (a \le s, t \le b)$$

Proof. It suffices to prove Lemma 4 for $a = s = 0$. The operator $V_2(t) = W_2(t, 0)$ is a solution of the impulsive operator system

$$\frac{dV_2}{dt} - A_1 W_2 = (A_2 - A_1) W_2\big|_{t \neq t_n}$$

$$V_2(t_n+0) = Q_n V(t_n)$$

$$V_2(0) = I$$

where I is the identity map in X.

Consider the impulsive operator system

$$\frac{dX}{dt} - A_1 X = F(t)\big|_{t \neq t_n}$$

$$X(t_n+0) = Q_n X(t_n)$$

$$X(0) = I$$

where $F(t) = [A_2(t) - A_1(t)] V_2(t)$. Its solution in view of (12) is given by the formula

$$V_2(t) = W_1(t, 0) + \int_0^t W_1(t, \tau)[A_2(\tau) - A_1(\tau)] V_2(\tau) \, d\tau$$

We set $\varphi(t) = \| V_2(t) \|$ and obtain

$$\varphi(t) \le N e^{-\nu_1(t-s)} + N \int_0^t e^{-\nu_1(t-\tau)} p(\tau) \varphi(\tau) \, d\tau$$

where $p(t) = \|A_2(t) - A_1(t)\|$.

The proof of Lemma 4 follows from Lemma 3. ■

Theorem 8. Let the following conditions be satisfied:

1. The operators Q_n $(n=1,2,\ldots)$ are invertible.
2. The operator-valued function $A(t)$ is compact and the spectra of all ω-limiting operators of $A(t)$ lie in the half-plane $\operatorname{Re}\lambda < -\nu_0(\nu_0>0)$.
3. The operator-values function $A(t)$ satisfies coneition $S_{\varepsilon,L}$ for $\varepsilon>0$ small enough and L large enough and the numbers ε and L depend only on the set of the ω-limiting operators of $A(t)$.
4. For $n=1,2,\ldots$, the inequality $\|Q_n\|\leq 1$ holds. Then the general exponent κ_g of the impulsive equation (1), (2) is negative.

Proof. In view of (8) and conditions (1) and (3) of Theorem 8, we have $\kappa_g<\infty$. From condition 2 of Theorem 8 it follows that for τ large enough the following inequality holds:

$$\|A(t)-A(\tau)\|<\varepsilon \qquad (\tau\leq t\leq \tau+L)$$

From Lemma 6.3 of Daleckii and Krein (1974) it follows that there exists a number $T_0>0$ such that for $\tau>T_0$ we have

$$\|e^{A(\tau)t}\|\leq N_0\, e^{-\nu_0 t} \tag{25}$$

where N_0 and ν are constants. We set $a=\tau$, $b=\tau+L$, $A_1(t)\equiv A(\tau)$, $A_2(t)=A(t)$ $(\tau\leq t\leq\tau+L)$. Let $W(t,s)$ and $W_1(t,s)$ be the respective evolutionary operators of the impulsive equation (1), (2) with operator-valued functions $A(t)$ and $A_1(t)$. For $\|W_1(t,s)\|$ for $t\in(t_n,t_{n+1}]$, $s\in(t_{k-1},t_k]$, $k\leq n$, the following estimate holds:

$$\begin{aligned}\|W_1(t,s)\| &= \|U(t,t_n)Q_nU(t_n,t_{n-1})Q_{n-1}\cdots Q_{k+1}U(t_{k+1},t_k)\\ &\quad Q_kU(t_k,s)\| = \|e^{A(\tau)(t-t_n)}Q_n\, e^{A(\tau)(t_n-t_{n-1})}Q_{n-1}\cdots Q_{k+1}\, e^{A(\tau)(t_{k+r}-t_k)}\\ &\quad Q_k\, e^{A(\tau)(t_k-s)}\|\leq N_0\, e^{-\nu_0(t-s)}\end{aligned}$$

By Lemma 4 for $\tau\leq s\leq t\leq\tau+L$ we obtain the following estimate for $\|W(t,s)\|$:

$$\begin{aligned}\|W(t,s)\| &\leq N_0\exp\left[N_0\int_s^t\|A(r)-A(\tau)\|\,dr\right]\\ &\leq N_0\exp[-(\nu_0-N_0\varepsilon)(t-s)]\\ &= N_0\exp[-\nu(t-s)]\end{aligned}$$

where $\nu=\nu_0-N_0\varepsilon$. Choose $\varepsilon<\nu_0/N_0$, i.e., $\nu>0$. Then $\|W(\tau+L,\tau)\|\leq N_0\, e^{-\nu L}$. Choose $L>(\ln N_0)/(\nu_0-N_0\varepsilon)$ and obtain $\|W(\tau+L,\tau)\|\leq q<1$, where $q=N_0\, e^{-\nu L}$.

The assertion of Theorem 8 follows from the fact that $\kappa_g < \infty$ and from Theorem 6. ■

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